

Remark on the coherent information saturating its upper bound

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Abstract

Coherent information is a useful concept in quantum information theory. It connects with other notions in data processing. In this short remark, we discuss the coherent information saturating its upper bound. A necessary and sufficient condition for this saturation is derived.

1 Coherent information inequality

The fundamental problem in quantum error correction is to determine when the effect of a quantum channel (trace-preserving completely positive map) $\Phi \in \mathcal{T}(\mathcal{H}_B)$ acting on half of a pure entangled state can be perfectly reversed. Define the *coherent information*

$$I_c(\rho, \Phi) \stackrel{\text{def}}{=} S(\Phi(\rho)) - S(\mathbb{1}_A \otimes \Phi(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|)), \quad (1.1)$$

where $|\mathbf{u}_\rho\rangle = \sum_j \sqrt{\lambda_j} |x_j\rangle \otimes |\lambda_j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is any *purification* of $\rho = \sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j|$.

In general, we have

$$I_c(\rho, \Phi) \leq S(\rho). \quad (1.2)$$

It was shown that there exists a quantum channel Ψ (see [1]) such that

$$I_c(\rho, \Phi) = S(\rho) \iff (\mathbb{1}_A \otimes \Psi \circ \Phi)(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|) = |\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|. \quad (1.3)$$

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By the *Stinespring dilation theorem*, we may assume that

$$\Phi(\rho) = \text{Tr}_C(U(\rho \otimes |\epsilon\rangle\langle\epsilon|)U^\dagger), \quad U \in \mathcal{U}(\mathcal{H}_B \otimes \mathcal{H}_C), |\epsilon\rangle \in \mathcal{H}_C,$$

which indicates that

$$\begin{aligned} \mathbb{1}_A \otimes \Phi(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|) &= \text{Tr}_C((\mathbb{1}_A \otimes U)(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho| \otimes |\epsilon\rangle\langle\epsilon|)(\mathbb{1}_A \otimes U)^\dagger) \\ &= \text{Tr}_C(|\Omega\rangle\langle\Omega|), \end{aligned} \tag{1.4}$$

where $|\Omega\rangle = (\mathbb{1}_A \otimes U)(|\mathbf{u}_\rho\rangle \otimes |\epsilon\rangle)$. Now

$$|\Omega\rangle\langle\Omega| = (\mathbb{1}_A \otimes U)(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho| \otimes |\epsilon\rangle\langle\epsilon|)(\mathbb{1}_A \otimes U)^\dagger$$

is a tripartite state in $\mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$, it follows that

$$\begin{aligned} \text{Tr}_C(|\Omega\rangle\langle\Omega|) &= \mathbb{1}_A \otimes \Phi(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|) \equiv \Omega_{AB}, \\ \text{Tr}_A(|\Omega\rangle\langle\Omega|) &= U(\rho \otimes |\epsilon\rangle\langle\epsilon|)U^\dagger \equiv \Omega_{BC}, \\ \text{Tr}_{AC}(|\Omega\rangle\langle\Omega|) &= \Phi(\rho) \equiv \Omega_B, \end{aligned}$$

where $\Omega_{ABC} \equiv |\Omega\rangle\langle\Omega|$. From the above expressions, it is obtained that

$$\begin{aligned} S(\Omega_{ABC}) &= 0, \\ S(\Omega_B) &= S(\Phi(\rho)) \\ S(\Omega_{BC}) &= S(\rho), \\ S(\Omega_{AB}) &= S((\mathbb{1}_A \otimes \Phi)(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|)) \end{aligned}$$

Apparently, $I_c(\rho, \Phi) = S(\rho) \iff S(\Phi(\rho)) = S((\mathbb{1}_A \otimes \Phi)(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|)) + S(\rho)$, that is,

$$\begin{aligned} I_c(\rho, \Phi) = S(\rho) &\iff S(\Omega_B) = S(\Omega_{AB}) + S(\Omega_{BC}) \\ &\iff S(\Omega_B) - S(\Omega_C) = S(\Omega_{BC}). \end{aligned}$$

It follows from Proposition 2.2 in Appendix that this equation holds if and only if

- (i) \mathcal{H}_B can be factorized into the form $\mathcal{H}_B = \mathcal{H}_L \otimes \mathcal{H}_R$,
- (ii) $\Omega_{BC} = \rho_L \otimes |\psi\rangle\langle\psi|_{RC}$ for $|\psi\rangle_{RC} \in \mathcal{H}_R \otimes \mathcal{H}_C$.

Hence

$$U(\rho \otimes |\epsilon\rangle\langle\epsilon|)U^\dagger = \rho_L \otimes |\psi\rangle\langle\psi|_{RC} \implies \rho \otimes |\epsilon\rangle\langle\epsilon| = U^\dagger(\rho_L \otimes |\psi\rangle\langle\psi|_{RC})U.$$

Clearly, $\Omega_{ABC} = |\phi\rangle\langle\phi|_{AL} \otimes |\psi\rangle\langle\psi|_{RC}$. Thus

$$\begin{aligned} |\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho| &= (\mathbb{1}_A \otimes \Psi \circ \Phi)(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|) = (\mathbb{1}_A \otimes \Psi)(\Omega_{AB}) \\ &= (\mathbb{1}_A \otimes \Psi)(|\phi\rangle\langle\phi|_{AL} \otimes \rho_R). \end{aligned}$$

Since $|\Omega\rangle\langle\Omega| = (\mathbb{1}_A \otimes U)(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho| \otimes |\epsilon\rangle\langle\epsilon|)(\mathbb{1}_A \otimes U)^\dagger$, it follows that

$$\begin{aligned} |\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho| &= \text{Tr}_C \left((\mathbb{1}_A \otimes U)^\dagger |\Omega\rangle\langle\Omega| (\mathbb{1}_A \otimes U) \right) \\ &= \text{Tr}_C \left((\mathbb{1}_A \otimes U)^\dagger (|\phi\rangle\langle\phi|_{AL} \otimes |\psi\rangle\langle\psi|_{RC}) (\mathbb{1}_A \otimes U) \right). \end{aligned} \quad (1.5)$$

The above equation gives that

$$(\mathbb{1}_A \otimes \Psi)(|\phi\rangle\langle\phi|_{AL} \otimes \rho_R) = \text{Tr}_C \left((\mathbb{1}_A \otimes U)^\dagger (|\phi\rangle\langle\phi|_{AL} \otimes |\psi\rangle\langle\psi|_{RC}) (\mathbb{1}_A \otimes U) \right).$$

Given the state $\Omega_{AB} = \mathbb{1}_A \otimes \Phi(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|)$, the recovery procedure Ψ is:

- (i) preparing the state $|\psi\rangle_{RC}$ on $\mathcal{H}_R \otimes \mathcal{H}_C$; thus we have a state $|\phi\rangle\langle\phi|_{AL} \otimes |\psi\rangle\langle\psi|_{RC}$.
- (ii) next performing U^\dagger ; we get

$$(\mathbb{1}_A \otimes U)^\dagger (|\phi\rangle\langle\phi|_{AL} \otimes |\psi\rangle\langle\psi|_{RC}) (\mathbb{1}_A \otimes U).$$

- (iii) finally discarding the fixed ancillary state $|\epsilon\rangle\langle\epsilon|$;

$$\text{Tr}_C \left((\mathbb{1}_A \otimes U)^\dagger (|\phi\rangle\langle\phi|_{AL} \otimes |\psi\rangle\langle\psi|_{RC}) (\mathbb{1}_A \otimes U) \right).$$

Note that $\mathbb{1}_A \otimes \Phi(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|) = |\phi\rangle\langle\phi|_{AL} \otimes \rho_R$ implies that

$$\Phi(\rho) = \rho_L \otimes \rho_R.$$

This indicates that the coherent information reaches its maximal value if and only if the output state of the quantum channel Φ is a product state. Therefore we have the following theorem:

Theorem 1.1. *Let $\rho \in \mathcal{D}(\mathcal{H})$ and $\Phi \in \mathcal{T}(\mathcal{H})$ be a quantum channel. The coherent information achieves its maximum, that is, $I_c(\rho, \Phi) = S(\rho)$ if and only if the following statements holds:*

- (i) *the underlying Hilbert space can be decomposed as: $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$;*
- (ii) *the output state of the quantum channel Φ is of a product form: $\Phi(\rho) = \rho_L \otimes \rho_R$ for $\rho_L \in \mathcal{D}(\mathcal{H}_L), \rho_R \in \mathcal{D}(\mathcal{H}_R)$.*

Remark 1.2. Consider a Kraus representation of a quantum channel $\Phi \in \mathcal{T}(\mathcal{H})$ in its canonical Kraus form: $\Phi = \sum_k \text{Ad}_{M_k}$. For any $\rho \in \mathcal{D}(\mathcal{H})$, define

$$\widehat{\Phi}(\rho) \stackrel{\text{def}}{=} \sum_{i,j} \text{Tr}(M_i \rho M_j^\dagger) |i\rangle\langle j|.$$

If ρ is purified as $|\mathbf{u}_\rho\rangle \in \mathcal{H} \otimes \mathcal{K}$ with $\dim(\mathcal{K}) \geq \dim(\mathcal{H})$, then

$$\mathcal{S}(\widehat{\Phi}(\rho)) = \mathcal{S}\left((\Phi \otimes \mathbb{1}_{\mathcal{L}(\mathcal{K})})(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|)\right).$$

Indeed, let $\rho = \sum_k \lambda_k |\lambda_k\rangle\langle\lambda_k|$ be its spectral decomposition,

$$\begin{aligned} |\mathbf{u}_\rho\rangle &\stackrel{\text{def}}{=} \sum_k \sqrt{\lambda_k} |\lambda_k\rangle \otimes |\lambda_k\rangle, \\ |\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho| &= \sum_{m,n} \sqrt{\lambda_m \lambda_n} |\lambda_m\rangle\langle\lambda_n| \otimes |\lambda_m\rangle\langle\lambda_n| \\ |\Omega\rangle &\stackrel{\text{def}}{=} \sum_{k,i} \sqrt{\lambda_k} M_i |\lambda_k\rangle \otimes |\lambda_k\rangle \otimes |i\rangle. \end{aligned}$$

Thus

$$|\Omega\rangle\langle\Omega| = \sum_{m,n,i,j} \sqrt{\lambda_m \lambda_n} M_i |\lambda_m\rangle\langle\lambda_n| M_j^\dagger \otimes |\lambda_m\rangle\langle\lambda_n| \otimes |i\rangle\langle j|,$$

which implies that

$$\begin{aligned} \text{Tr}_3(|\Omega\rangle\langle\Omega|) &= \sum_{m,n,i} \sqrt{\lambda_m \lambda_n} M_i |\lambda_m\rangle\langle\lambda_n| M_i^\dagger \otimes |\lambda_m\rangle\langle\lambda_n| \\ &= \Phi \otimes \mathbb{1}_{\mathcal{L}(\mathcal{K})}(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|), \\ \text{Tr}_{1,2}(|\Omega\rangle\langle\Omega|) &= \sum_{i,j} \text{Tr}(M_i \rho M_j^\dagger) |i\rangle\langle j| = \widehat{\Phi}(\rho). \end{aligned}$$

Clearly, $\mathcal{S}\left((\Phi \otimes \mathbb{1}_{\mathcal{L}(\mathcal{K})})(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|)\right)$ is independent of an arbitrary purification $|\mathbf{u}_\rho\rangle$ of ρ .

In fact, if $|\mathbf{u}_\rho^{(1)}\rangle$ and $|\mathbf{u}_\rho^{(2)}\rangle$ are any two purification of ρ , then by Schimdt decomposition:

$$\begin{aligned} |\mathbf{u}_\rho^{(1)}\rangle &= \sum_k \sqrt{\lambda_k} |\lambda_k\rangle \otimes |x_k\rangle, \\ |\mathbf{u}_\rho^{(2)}\rangle &= \sum_k \sqrt{\lambda_k} |\lambda_k\rangle \otimes |y_k\rangle, \end{aligned}$$

it is seen that there exists an isometry operator U such that $U|x_k\rangle = |y_k\rangle$ for each k , moreover $|\mathbf{u}_\rho^{(2)}\rangle = (\mathbb{1} \otimes U)|\mathbf{u}_\rho^{(1)}\rangle$. Now $|\mathbf{u}_\rho^{(2)}\rangle\langle\mathbf{u}_\rho^{(2)}| = (\mathbb{1} \otimes U)|\mathbf{u}_\rho^{(1)}\rangle\langle\mathbf{u}_\rho^{(1)}|(\mathbb{1} \otimes U)^\dagger$, which implies that

$$\begin{aligned} (\Phi \otimes \mathbb{1})(|\mathbf{u}_\rho^{(2)}\rangle\langle\mathbf{u}_\rho^{(2)}|) &= (\mathbb{1} \otimes U)(\Phi \otimes \mathbb{1})(|\mathbf{u}_\rho^{(1)}\rangle\langle\mathbf{u}_\rho^{(1)}|)(\mathbb{1} \otimes U)^\dagger, \\ \mathcal{S}\left((\Phi \otimes \mathbb{1})(|\mathbf{u}_\rho^{(1)}\rangle\langle\mathbf{u}_\rho^{(1)}|)\right) &= \mathcal{S}\left((\Phi \otimes \mathbb{1})(|\mathbf{u}_\rho^{(2)}\rangle\langle\mathbf{u}_\rho^{(2)}|)\right). \end{aligned}$$

2 Appendix

2.1 The saturation of the strong subadditivity inequality

Proposition 2.1 ([1]). *A state $\rho_{ABC} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ saturating the strong subadditivity inequality, i.e.,*

$$S(\rho_{AB}) + S(\rho_{BC}) = S(\rho_{ABC}) + S(\rho_B)$$

if and only if there is a decomposition of system B as

$$\mathcal{H}_B = \bigoplus_j \mathcal{H}_{b_j^L} \otimes \mathcal{H}_{b_j^R}$$

into a direct (orthogonal) sum of tensor products, such that

$$\rho_{ABC} = \bigoplus_j \lambda_j \rho_{Ab_j^L} \otimes \rho_{b_j^R C},$$

where $\rho_{Ab_j^L} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_{b_j^L})$ and $\rho_{b_j^R C} \in \mathcal{D}(\mathcal{H}_{b_j^R} \otimes \mathcal{H}_C)$, and $\{\lambda_j\}$ is a probability distribution.

2.2 The saturation of Araki-Lieb inequality

The following proposition can be seen as a characterization of the saturation of Araki-Lieb inequality:

$$|S(\rho_B) - S(\rho_C)| \leq S(\rho_{BC}). \quad (2.1)$$

For the readers' convenience, we copy the proof here.

Proposition 2.2 ([2]). *Let $\rho_{BC} \in \mathcal{D}(\mathcal{H}_B \otimes \mathcal{H}_C)$. The reduced states are $\rho_B = \text{Tr}_C(\rho_{BC})$, $\rho_C = \text{Tr}_B(\rho_{BC})$, respectively. Then $S(\rho_{BC}) = S(\rho_B) - S(\rho_C)$ if and only if*

(1) \mathcal{H}_B can be factorized into the form $\mathcal{H}_B = \mathcal{H}_L \otimes \mathcal{H}_R$,

(2) $\rho_{BC} = \rho_L \otimes |\psi\rangle\langle\psi|_{RC}$ for $|\psi\rangle_{RC} \in \mathcal{H}_R \otimes \mathcal{H}_C$.

Proof. The sufficiency of the condition is immediate. The proof of necessity is presented as follows: Assume that $S(\rho_{BC}) = S(\rho_B) - S(\rho_C)$. The bipartite state ρ_{BC} can be purified

into a tripartite state $|\Omega_{ABC}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, where \mathcal{H}_A is a reference system. Denote $\rho_{ABC} = |\Omega_{ABC}\rangle\langle\Omega_{ABC}|$. We have

$$\begin{aligned}\text{Tr}_{AB}(\rho_{ABC}) &= \rho_C, & \text{Tr}_{AC}(\rho_{ABC}) &= \rho_B, \\ \text{Tr}_C(\rho_{ABC}) &= \rho_{AB}, & \text{Tr}_A(\rho_{ABC}) &= \rho_{BC}.\end{aligned}$$

Now since $S(\rho_{ABC}) = 0$, it follows that $S(\rho_C) = S(\rho_{AB})$. Thus we have

$$S(\rho_{AB}) + S(\rho_{BC}) = S(\rho_B) = S(\rho_B) + S(\rho_{ABC}),$$

which, by Proposition 2.1, implies that

- (i) \mathcal{H}_B can be factorized into the form $\mathcal{H}_B = \bigoplus_{k=1}^K \mathcal{H}_{b_k^L} \otimes \mathcal{H}_{b_k^R}$,
- (ii) $\rho_{ABC} = \bigoplus_{k=1}^K \lambda_k \rho_{Ab_k^L} \otimes \rho_{b_k^R C}$ for $\rho_{Ab_k^L} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_{b_k^L})$ and $\rho_{b_k^R C} \in \mathcal{D}(\mathcal{H}_{b_k^R} \otimes \mathcal{H}_C)$, where $\{\lambda_k\}$ is a probability distribution.

Clearly,

$$S(\rho_{BC}) = S(\rho_B) - S(\rho_C) \implies S(\rho_A) + S(\rho_C) = S(\rho_{AC}).$$

But

$$S(\rho_A) + S(\rho_C) = S(\rho_{AC}) \iff \rho_{AC} = \rho_A \otimes \rho_C.$$

From the expression

$$\rho_{ABC} = \bigoplus_{k=1}^K \lambda_k \rho_{Ab_k^L} \otimes \rho_{b_k^R C},$$

it follows that

$$\rho_{AC} = \sum_{k=1}^K \lambda_k \rho_{A,k} \otimes \rho_{C,k}.$$

Combining all the facts above mentioned, we have

$$K = 1,$$

i.e., the statement (1) in the present theorem holds. Hence

$$\rho_{ABC} = \rho_{AL} \otimes \rho_{RC}$$

for $\rho_{AL} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_L)$ and $\rho_{RC} \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_C)$, which implies that both ρ_{AL} and ρ_{RC} are pure states since ρ_{ABC} is pure state. Therefore

$$\rho_{BC} = \text{Tr}_A(\rho_{AL}) \otimes \rho_{RC} = \rho_L \otimes |\psi\rangle\langle\psi|_{RC} \quad (2.2)$$

for $|\psi\rangle_{RC} \in \mathcal{H}_R \otimes \mathcal{H}_C$, i.e., the statement (2) holds. This completes the proof. \square

Remark 2.3. The result in Proposition 2.2 is employed to study the saturation of the upper bound of quantum discord in [3]. Later on, E.A Carlen gives an elementary proof about this result in [4].

References

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